Review Exercises for the MATH 151/161 Final Exam

Problems appearing on your in-class final will be similar to those here but will have numbers and functions changed. For example, if problem 1 were selected for your in-class final exam, it might look like this:

1. If f is a continuous function and $\csc(3x)-1=\int_{\pi/6}^{x} f(t)dt$, compute the exact value of $f\left(\frac{\pi}{12}\right)$. Simplify your answer as much as possible.

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1. If f is a continuous function and $\cot(2x)-1=\int_{\pi/8}^{x} f(t)dt$, compute the exact value of $f\left(\frac{\pi}{8}\right)$. Simplify your answer as much as possible.

2. Give an example of each of the following, or briefly explain why none exists:

   a. A function $y=f(x)$ for which:

      i. $dy=\Delta y$, for all x.

      ii. $dy>\Delta y$, for all x

   b. A function $y=g(x)$ which is differentiable, but ***not*** continuous, at the domain point $x=2$.

   c. A function $y=h(x)$, continuous for all real numbers, which ***fails*** to have a derivative at precisely 3 of its domain values.

3. Evaluate:

   a. $\lim\limits_{x\to 3/2} \dfrac{4x^2-9}{8x^3-27}$

   b. $\lim\limits_{x\to\infty} \dfrac{3x^2+2\sin(2x)}{x^2+1}$

   c. $\lim\limits_{x\to\infty} \dfrac{2x^3+4x^2+7}{5x^3+2x^2+6}$

   d. $\lim\limits_{x\to\infty} \left(4+2e^{-x}\right)$

   e. $\lim\limits_{x\to 3^-} \dfrac{x+4}{x-3}$

f. $\lim_{x\to\left(-\frac{\pi}{2}\right)^+} \tan(x)$

g. $\lim_{x\to\infty} \text{Arctan}(x)$

h. $\lim_{x\to 0} \dfrac{3^x + 3^{-x} - 2}{1-\cos(2x)}$

i. $\lim_{x\to\infty} \left(1+\dfrac{a}{x}\right)^{bx}$

j. $\lim_{x\to\infty} \dfrac{\sqrt{4x+1}}{\sqrt{x+1}}$

k. Let $S(x) = \int_0^x \sin\left(\dfrac{\pi t^2}{2}\right) dt$, which arises in the study of the diffraction of light waves. Evaluate $\lim_{x\to 0} \dfrac{S(x)}{x^3}$.

4. Find $\dfrac{dy}{dx}$ for each of the following:

   a. $x^4 + e^{xy} - y^2 = 20$

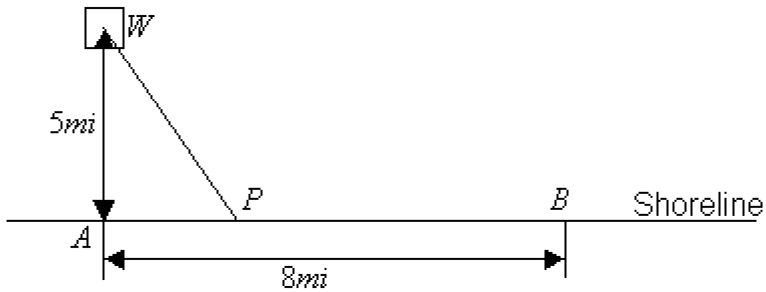
   b. $y = \left(4x^2 - 7\right)^{2+\sqrt{x^2-5}}$

   c. $y = \sin^2(3x) \cdot \cos\left(3x^3 + 5\right)$

   d. $y = \sin(t)$, $x = t^2 + 1$. (Leave answer in terms of t.)

   e. $y = \int_{x^2}^{x^3} \sqrt{1+t^2}\, dt$

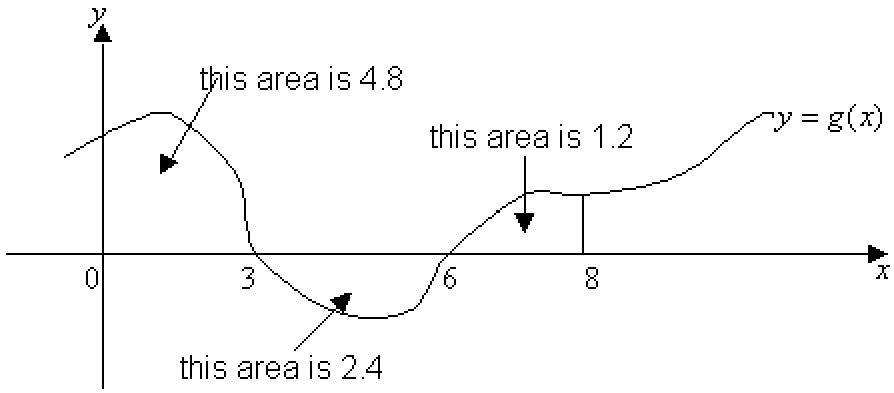
5. An offshore oil well is located in the ocean at a point W, which is 5 mi from the closest shore point A on a straight shoreline. The oil is to be piped to a shore point B that is 8 mi from A by piping it on a straight line under water from W to some shore point P between A and B and then on to B via a pipe along the shoreline. If the cost of laying pipe is \$100,000 per mile under water and \$75,000 per mile over land, where should the point P be located to minimize the cost of laying the pipe?



6. a. Find both y' and y'' **in terms of *x* and *y* only** from the equation $y^2 + xy = 3$.

   b. Evaluate y' and y'' when $x = 2$ and $y = 1$. $y' =$ ____________; $y'' =$ ____________

7. Given the areas as shown for the graph of the continuous function $y = g(x)$, evaluate each of the following integrals.



a. $\int_0^4 5|g(2x)|\,dx$

b. $\int_{1/2}^1 x\, g\left(4x^2 - 1\right) dx$

c. $\int_0^{\pi/4} \frac{g\left(3\cos(2x)\right)}{\csc(2x)}\,dx$

d. $\int_{\pi/18}^{\pi 6} \csc(3x) g(3\csc(3x)) \cot(3x)\,dx$

e. $\int_{\frac{1}{2}\ln 6}^{\frac{1}{2}\ln 8} g\left(e^{2x}\right) e^{2x} dx$

8.

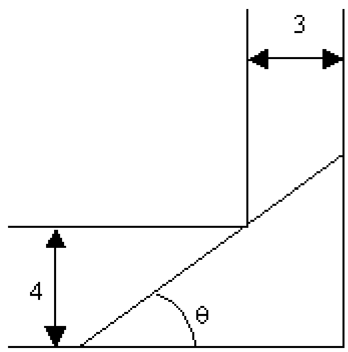
a. State the mean value theorem for a function on a closed interval $a \le x \le b$ and illustrate with a simple sketch.

b. Given $f(x) = 2x^{2/3}, a = -1,$ and $b = 1$, show that there is no number c between a and b satisfying the conclusion of the mean value theorem.

c. Explain why the mean value theorem does <u>not</u> apply to the function given in part b.

9. Suppose that $f(x)$ and $g(x)$ are differentiable functions for $a \le x \le b$ such that $f'(x) = g'(x)$ for every x in the open interval $a < x < b$. Prove that: $f(b) - f(a) = g(b) - g(a)$.

10. Two corridors 3 feet wide and 4 feet wide, respectively, meet at a right angle. Approximate the length of the longest non-bendable rod that can be carried horizontally around the corner, as shown in the sketch. Disregard the thickness of the rod. Round your answers to two decimal places.



11. Let $y = f(x) = \dfrac{3}{2x-5}$; compute the following:

a. $f(x + \Delta x)$

b. $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

c. $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

12. Given that $\int f'(u)\,du = f(u) + C$, express each of the following integrals in terms of the function f.

a. $\int x\, f'(3x^2 + 1)\,dx$

b. $\int \dfrac{f'(2/x)\,dx}{x^2}$

c. $\int f'(\sec(3x))\cdot\sec(3x)\cdot\tan(3x)\,dx$

13. A particle is traveling upward and to the right along the curve $y=\ln(1-\cot(x))$. Its x-coordinate is increasing at the rate $\frac{dx}{dt}=\sin(x)$ m/sec. At what rate is the y-coordinate changing at the point $\left(\frac{3\pi}{4},\ln 2\right)$?

14. Suppose a shoreline has the shape of the parabola $y=\frac{1}{5}x^2$ where x and y are measured in miles, and that a fog light located at $(0,2)$ revolves at the rate of $\frac{1}{2}$ radian per second. How fast does the x-coordinate of the point of illumination on the shoreline change at the instant the point $\left(1,\frac{1}{5}\right)$ is illuminated?

15. Sketch a graph of the curve $y=g(x)$ from $x=-5$ to $x=5$. Points $(2,1)$ and $(4,0)$ are on the graph of the function; the function has origin symmetry. You are also given that $x=\pm 3$ are asymptotes, that $g''(x)<0$ if $x>3$ and that:

$g'(x)>0$ if $|x|<2$
$g'(x)=0$ if $x=\pm 2$ or ± 4
$g'(x)<0$ if $2<|x|<3$

16. A cylindrical tin boiler of given volume V_0 has a copper bottom and is open at the top. If sheet copper is 5 times as expensive as sheet tin per unit area, find the most economical dimensions (height and radius) for constructing the boiler.

17. Suppose f, g and h are differentiable at $x=3$ and that $f(3)=2, g(3)=4, h(3)=1, f'(3)=6, g'(3)=5,$ and $h'(3)=\frac{1}{2}$. Find $\left[\frac{dy}{dx}\right]_{x=3}$ if y is as follows. Simplify your answer.

a. $y=\dfrac{\sqrt{f^3(x)\cdot g^{\frac{1}{2}}(x)}}{h^5(x)}$

b. $y=f(x)^{g(x)}$

18. Two moving particles have acceleration (at time t seconds) given by $a_1=4t+4$ and $a_2=\frac{2t}{3}+5$. Assuming that both particles start from rest at $t=0$, do they ever again have the same velocity? If so, when?

19.

a. Define what it means for a function $f(x)$ to be continuous at $x = a$ (where a is a real number).

b. Find the values of the constants a and b so that $y = f(x)$ is continuous, and draw a sketch of its graph.

$$f(x) = \begin{cases} 1 & \text{if } x \le 3 \\ ax + b & \text{if } 3 < x < 5 \\ 7 & \text{if } x \ge 5 \end{cases}$$

20. Consider the function $y = F(x)$ defined by $F(x) = \int_0^x e^{-t^2}\,dt$.

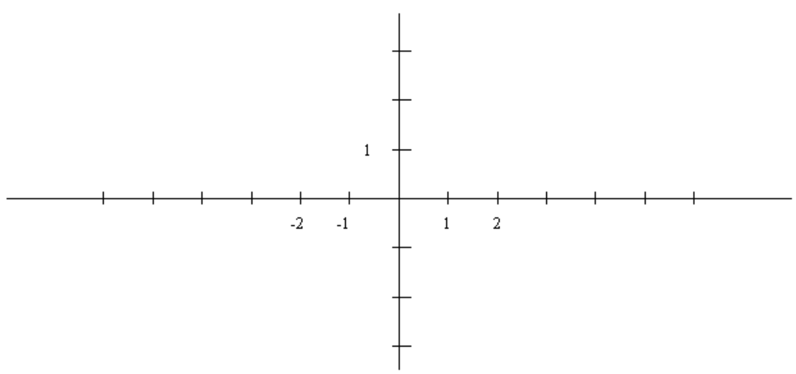
a. Evaluate the following: $F(0),\ F'(0),\ F''(0)$.

b. Find all maxima, minima, and inflection points of $y = F(x)$. Over what interval is the graph of $y = F(x)$ concave downward?

c. Use your calculator to approximate the values $F(1)$, $F(2)$, and $F(3)$.

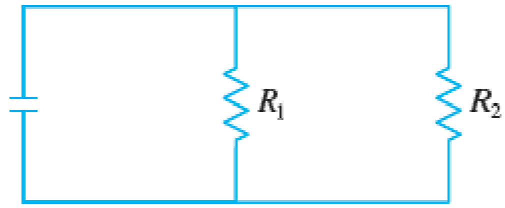
$F(1) \approx$
$F(2) \approx$
$F(3) \approx$

d. Use parts a., b., and c. and any symmetry the function may possess to make a careful sketch of the curve $y = F(x)$ on the interval: $-3 \le x \le 3$.



21. Solve for y in terms of x. if: $\dfrac{d^2y}{dx^2} = 6 - \dfrac{12}{x^5} + \pi^2\cos(\pi x)$ and $\dfrac{dy}{dx} = 10,\ y = 2,$ when $x = 1$.

22. If two resistors with resistances R_1 and R_2 are connected in parallel, as in the figure below, then the total resistance R, measured in ohms (Ω), is given by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$.



If R_1 and R_2 are increasing at rates of $0.3\ \Omega/s$ and $0.2\ \Omega/s$, respectively, how fast is R changing when $R_1 = 60\Omega$ and $R_2 = 80\Omega$? (Round your answer to three decimal places.)

23. A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft. (Round your answers to three decimal places.)

   a. How fast is the distance from the television camera to the rocket changing at that moment?

   b. If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

24. Determine whether the statement is true or false. **Briefly justify your answers**.

Statements	True	False
a) If $\lim_{x\to 3} f(x) = 5$, then $f(3) = 5$.		
b) If $f'(x)$ exists and is nonzero for all x, then $f(1) \neq f(0)$.		
c) If $f(1) = 2$ and $f(3) = 5$, then there exists a number c between 1 and 3 such that $f(c) = 4$.		
d) If f is continuous at 5 and $f(5) = -1$ and $f(4) = 3$, then $\lim_{x\to 2} f\left(4x^2 - 11\right) = -1$.		
e) If f is differentiable at a, then f is continuous at a.		
f) If f is continuous at a, then f is differentiable at a.		

Statements	True	False
g) If f is differentiable and $f(-2)=f(2)$, then there is a number c such that $\lvert c\rvert<2$ and $f'(c)=0$.		
h) There exists a function f such that $f(x)>0$, $f'(x)<0$, and $f''(x)>0$ for all x.		
i) If $g(x)=x^5$, then $\lim_{x\to 3}\frac{g(x)-g(3)}{x-3}=405$.		
j) If $\int_0^3 f(x)\,dx=\int_0^3 g(x)\,dx$, then $f(x)=g(x)$		
k) If $f'(x)=g'(x)$, then: $f(x)=g(x)$.		
l) $\int_0^1 2x\,f(x)\,dx=\int_0^{\frac{\pi}{2}} f(\sin x)\sin(2x)\,dx$		
m) If $\int f(x)dx=\int g(x)dx$, then $f(x)=g(x)$		

25. Evaluate the following integrals. Show the details of substitutions you make in reaching your answers.

a. $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

b. $\int_{-1}^2 \frac{x}{\sqrt{x+2}}dx$

c. $\int_{-2}^1 \lvert 2x+1\rvert dx$

d. $\int_0^1 \frac{e^x}{e^x+1}dx$

e. $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}dx$

f. $\int_0^{\frac{1}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$

ANSWER KEY

1. $f\left(\frac{\pi}{8}\right) = -4$

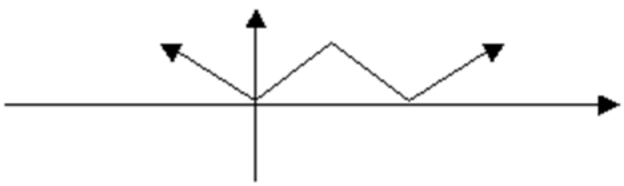
2.

a. (i) Any constant will do, for example: $y = 1$. (ii) Any functions that concave down will do, for example: $y = -x^2$.

b. Impossible: we know that at any point where $\frac{dy}{dx}$ is finite, the function must also be continuous.

c. How about this function:

$$f(x) = \begin{cases} -x, & \text{for } x < 0 \\ x, & \text{for } 0 \le x \le 1 \\ -x+2, & \text{for } 1 \le x \le 2 \\ x-2, & \text{for } 2 \le x \end{cases}$$



3.

a. $\frac{2}{9}$

b. 3

c. $\frac{2}{5}$

d. 4

e. $-\infty$

f. $-\infty$

g. $\frac{\pi}{2}$

h. $\frac{(\ln 3)^2}{2}$

i. e^{ab}

j. 2

k. $\frac{\pi}{6}$

4.

a. $\frac{dy}{dx} = \frac{-4x^3 - ye^{xy}}{xe^{xy} - 2y}$

b. $\frac{dy}{dx} = y\left[\frac{\left(2+\sqrt{x^2-5}\right)8x}{4x^2-7} + \frac{x\ln(4x^2-7)}{\sqrt{x^2-5}}\right]$

c. $\frac{dy}{dx} = \sin^2(3x)\left[-\sin(3x^3+5)\right](9x^2) + \cos(3x^3+5)\left[2\sin(3x)\right]\left[\cos(3x)\right](3)$

d. $\frac{dy}{dx} = \frac{\cos(t)}{2t}$

e. $\frac{dy}{dx} = -\sqrt{1+x^4}\cdot 2x + \sqrt{1+x^6}\cdot 3x^2$

5. Let x be the distance from A to P. $x \approx 5.6695$ mi

6. a. $y' = \frac{-y}{x+2y}$, $y'' = \frac{2xy+2y^2}{(x+2y)^3}$

   b. $y' = -\frac{1}{4}$, $y'' = 3/32$

7.
   a. 21
   b. 0.6
   c. 0.8
   d. $-\frac{4}{15}$
   e. 0.6

8.
   a. Any function $f(x)$ that is continuous on the closed interval $a \le x \le b$ and is differentiable for $a < x < b$ must have at least one $x-$value c between a and b where $f'(c) = \frac{f(b)-f(a)}{b-a}$. See the figure 1 and 2 on page 272 of the textbook.

   b. Since $f'(c) = \frac{4}{3c^{1/3}}$ and $\frac{f(b)-f(a)}{b-a} = 0$ It is impossible for $\frac{4}{3c^{1/3}} = 0$, therefore there is no such c value.

   c. Since $f'(x) = \frac{4}{3x^{1/3}}$ is undefined at $x = 0$.

9. Proof:
   By definition, $f(x)$ is an antiderivative of $f'(x)$ and $g(x)$ is an antiderivative of $g'(x)$.
   From the information given: $f'(x) = g'(x)$, this implies $g(x)$ is also an antiderivative of $f'(x)$.
   According to T**heorem 1** in Section 4.8 of the textbook, *two antiderivatives of the same function differ only by a constant*.
   Thus: $f(x) - g(x) = C$, or $f(x) = g(x) + C$, where C is some constant.
   Hence at $x = a$ and $x = b$, we have: $\left.\begin{matrix} f(a) = g(a) + C \\ f(b) = g(b) + C \end{matrix}\right\} \Rightarrow f(b) - f(a) = g(b) - g(a)$.

10. $L \approx 9.87$ feet; $\theta \approx 47.74^\circ$ or 0.83 radians.

11.
   a. $\frac{3}{2(x+\Delta x)-5}$
   b. $\frac{-6}{\left[2(x+\Delta x)-5\right](2x-5)}$
   c. $-6(2x-5)^{-2}$

12.

a. $\frac{1}{6} f(3x^2+1)+C$

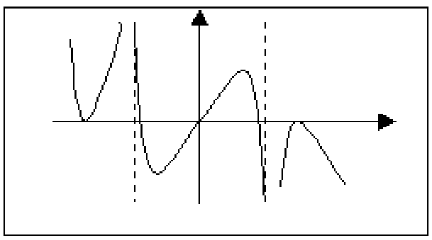
b. $-\frac{1}{2} f(2/x)+C$

c. $\frac{1}{3} f\left(\sec(3x)\right)+C$

13. $\frac{\sqrt{2}}{2}$ m/sec

14. 0.963636 unit/sec

15.



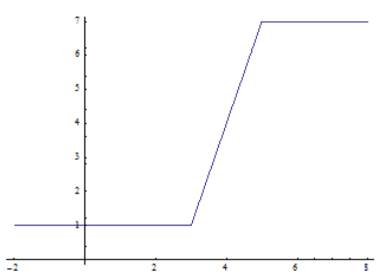
16. $r=\sqrt[3]{\frac{V_0}{5\pi}}$; and $h=\sqrt[3]{\frac{25V_0}{\pi}}$

17. a. $\frac{37}{4}$, b. $192+80\ln\left(2\right)$ $192+80\ln 2$

18. $t=\frac{3}{5}$ sec

19. a. $f(x)$ is continuous at a domain number a if $\begin{cases} i. f(a) \text{ is finite} \\ \text{ii. } \lim_{x\to a} f(x)=f(a) \end{cases}$

b. $a=3,\ b=-8$

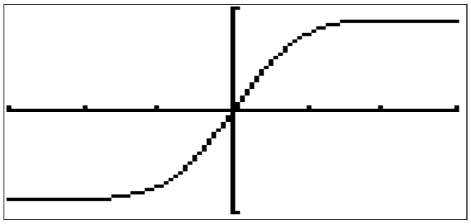


20. a. $F(0)=0, F'(0)=1,\ F''(0)=0$.

b. There is no maximum, and no minimum. $F(x)$ is concave downward for all $x>0$.

c. $F(1)\approx 0.75, F(2)\approx 0.88,\ F(3)\approx 0.89$.

d.



21. $y = 3x^2 - \frac{1}{x^3} - \cos(\pi x) + x - 2$.

22. $0.135\ \Omega/\text{s}$.

23. a. $360\ {}^{ft}/_{s}$ b. $0.096\ {}^{Rad}/_{s}$

24. Justification for True/False:

Statements	True	False
a) If $\lim_{x\to 3} f(x) = 5$, then $f(3) = 5$.		False. $f(x)$ may not be continuous at 3.
b) If $f'(x)$ exists and is nonzero for all x, then $f(1) \neq f(0)$.	True By the Mean Value Theorem.	
c) If $f(1) = 2$ and $f(3) = 5$, then there exists a number c between 1 and 3 such that $f(c) = 4$.		False f may not be continuous in the interval $[1,3]$.
d) If f is continuous at 5 and $f(5) = -1$ and $f(4) = 3$, then $\lim_{x\to 2} f\left(4x^2 - 11\right) = -1$.	True By the rule of the limit of a composite function.	
e) If f is differentiable at a, then f is continuous at a.	True Differentiable at a implies continuous at a.	
f) If is continuous at a, then is differentiable at a.		False f may have a sharp turn at a.
g) If f is differentiable and $f(-2) = f(2)$, then there is a number c such that $\lvert c \rvert < 2$ and $f'(c) = 0$.	True By the Mean Value Theorem.	

Statements	True	False
h) There exists a function f such that $f(x)>0,\ f'(x)<0,$ and $f''(x)>0$ for all x.	True e.g. $f(x)=e^{-x}$	
i) If $g(x)=x^5$, then $\lim_{x\to 3}\frac{g(x)-g(3)}{x-3}=405$.	True Hint: $\lim_{x\to 3}\frac{g(x)-g(3)}{x-3}=g'(3)$	
j) If $\int_0^3 f(x)\,dx=\int_0^3 g(x)\,dx$, then $f(x)=g(x)$		False Counter example: $\int_0^3 xdx=4.5$ $\int_0^3 1.5\,dx=4.5$
k) If $f'(x)=g'(x)$, then: $f(x)=g(x)$.		False Counter Example: $(x^2+2)'=(x^2)'$
l) $\int_0^1 2x\,f(x)\,dx=\int_0^{\frac{\pi}{2}} f(\sin x)\sin(2x)\,dx$ If , then	True Hint: Use u-substitution and a double angle formula.	
m) If $\int f(x)dx=\int g(x)dx$, then $f(x)=g(x)$	True Hint: $\frac{d}{dx}\left[\int f(x)dx\right]=f(x)$ $\frac{d}{dx}\left[\int g(x)dx\right]=g(x)$	

25. a. $2(e^2-e)$ b. $\frac{2}{3}$ c. $\frac{9}{2}=4.5$ d. $\ln\left(\frac{e+1}{2}\right)$ e. 2 f. $\frac{\pi^2}{72}$